Asymptotics

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Finite Sample Properties

Definition 1: Finite Sample Properties

The finite sample properties of an estimator describe how an estimator behaves in smaller samples.

- Under MLR Assumptions 1-6, the OLS estimator has the finite sample property

$$\widehat{\boldsymbol{\beta}} \sim \mathbb{N}\left(\boldsymbol{\beta}, \sigma^2 \left(X'X\right)^{-1}\right).$$

Asymptotics

Question 1

If MLR Assumptions 1-4 do not hold, then OLS isn't unbiased. If MLR Assumptions 1-6 do not hold, then our T and F-statistics don't necessarily follow t- and F-distributions so we can't do inference. In either case, are we screwed?

Asymptotics

Question 1

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Answer to Question 1

Depends on our sample size n (really $n - k$). If $n - k$ is sufficiently large, then asymptotic properties may save us!

Asymptotics

Definition 2: Asymptotics

Asymptotics refers to an estimators' large sample properties

- How an estimator behaves as the sample size n gets large?
  - Do we get closer to what we our estimating as $n \to \infty$?
  - Does our sampling distribution get tighter around the truth as $n \to \infty$?

Consistency

Definition 3: Consistency

An estimator $\widehat{\beta}$ is consistent for β if

$$\mathbb{P}\left(\left|\widehat{\beta}-\beta\right|>\epsilon\right)\to 0$$

as $n\to\infty$.

- You can think of ϵ as zero.
- As n gets large, $\widehat{\beta}$ gets close to β.
- We often write $\widehat{\beta}\xrightarrow{p}\beta$ or $\operatorname{plim}\widehat{\beta}=\beta$.

OLS Consistency

Theorem 1: OLS Consistency

Under MLR Assumptions 1-3 and $\mathbb{E}\left[X'\boldsymbol{u}\right] = 0$, the OLS estimator $\widehat{\boldsymbol{\beta}}$ is consistent for $\boldsymbol{\beta}$.

- No need to make assumptions regarding the variance or distribution of the error term.
- MLR Assumption 4 not needed; it is stronger and actually implies $\mathbb{E}[X'\boldsymbol{u}] = 0$.

OLS Consistency

Proof 1: OLS Consistency Part 1

We wish to show $\operatorname{plim}\widehat{\boldsymbol{\beta}} = \operatorname{plim}\boldsymbol{\beta}$. By MLR Assumptions 1 and 3,

$$\begin{aligned}\operatorname{plim}\widehat{\boldsymbol{\beta}} &= \operatorname{plim}\left(\boldsymbol{\beta} + (X'X)^{-1}X'\boldsymbol{u}\right) \\ &= \boldsymbol{\beta} + \operatorname{plim}\left((X'X)^{-1}X'\boldsymbol{u}\right).\end{aligned}$$

Multiplying and dividing the far right side of the equality be n gives

$$\operatorname{plim}\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta} + \operatorname{plim}\left[\left(\frac{1}{n}X'X\right)^{-1}\frac{1}{n}X'\boldsymbol{u}\right].$$

OLS Consistency

Proof 1: OLS Consistency Part 2

The probability limit of a product is the product of probability limits. Thus,

$$\text{plim}\,\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta} + \text{plim}\left[\left(\frac{1}{n}X'X\right)^{-1}\right] \cdot \text{plim}\left[\frac{1}{n}X'\boldsymbol{u}\right].$$

By the weak law of large numbers (WLLN), MLR Assumption 2, and the $\mathbb{E}[X'\boldsymbol{u}] = 0$ assumption,

$$\begin{aligned}\text{plim}\left[\frac{1}{n}X'\boldsymbol{u}\right] &= \mathbb{E}\left[X'\boldsymbol{u}\right] \\ &= \mathbf{0}.\end{aligned}$$

OLS Consistency

Proof 1: OLS Consistency Part 4

Hence,

$$\begin{aligned} \text{plim}\,\widehat{\boldsymbol{\beta}} &= \boldsymbol{\beta} + \text{plim}\left[\left(\frac{1}{n}X'X\right)^{-1}\right] \cdot \text{plim}\left[\frac{1}{n}X'\boldsymbol{u}\right] \\ &= \boldsymbol{\beta} + \mathbb{E}[X'X] \cdot \mathbf{0} \\ &= \boldsymbol{\beta}. \quad \square \end{aligned}$$

Hooray!

OLS Asymptotic Normality

Theorem 2: OLS Asymptotic Normality

Under MLR Assumptions 1-5, the OLS estimator has the asymptotically normal distribution of

$$\widehat{\boldsymbol{\beta}} \overset{a}{\sim} \mathbb{N}\left(\boldsymbol{\beta}, \widehat{\mathbb{V}}\left[\widehat{\boldsymbol{\beta}}\right]\right)$$

$$\frac{\widehat{\beta}_j - \beta_j}{\text{se}\left[\widehat{\beta}_j\right]} \overset{a}{\sim} \mathbb{N}\left(0, 1\right).$$

- So we can do inference with large enough samples even without assuming error normality!

OLS Asymptotic Normality

Proof 2: OLS Asymptotic Normality Part 1

Let's also make MLR Assumption 5 so we get a nicer expression for the variance.

$$\begin{aligned} \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} &= (X'X)^{-1}X'\boldsymbol{u} \\ \iff \sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) &= \sqrt{n}(X'X)^{-1}X'\boldsymbol{u} \\ \iff \sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) &= \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'\boldsymbol{u}. \end{aligned}$$

OLS Asymptotic Normality

Proof 2: OLS Asymptotic Normality Part 2

By the Central Limit Theorem,

$$\begin{aligned}\frac{1}{\sqrt{n}}X'\boldsymbol{u} &= \sqrt{n}\left(\frac{1}{n}X'\boldsymbol{u} - \mathbb{E}[X'\boldsymbol{u}]\right)\\ &= \sqrt{n}\left(\frac{1}{n}X'\boldsymbol{u}\right)\\ &\xrightarrow{d} \mathbb{N}\left(0, \sigma^2 X'X\right).\end{aligned}$$

Note that $\frac{1}{n}X'\boldsymbol{u} = \frac{1}{n}\sum_{i=1}^{n} x_{ij}u_i$ for each covariate $j = 1, \ldots, k$. Moreover, $\mathbb{V}\left[X'\boldsymbol{u}\right] = \mathbb{E}\left[X'\sigma^2 I_n X\right] = \mathbb{E}\left[\sigma^2 X'X\right] = \sigma^2\mathbb{E}[X'X]$.

OLS Asymptotic Normality

Proof 2: OLS Asymptotic Normality Part 3

Thus,

$$\begin{aligned}\sqrt{n}\left(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right) &\xrightarrow{d} \left(\frac{1}{n}X'X\right)^{-1}\mathbb{N}\left(0,\sigma^2\mathbb{E}[X'X]\right)\\ &\xrightarrow{d} \mathbb{N}\left(0,\sigma^2\mathbb{E}\left[X'X\right]^{-1}\mathbb{E}[X'X]\mathbb{E}\left[X'X\right]^{-1}\right)\\ &\xrightarrow{d} \mathbb{N}\left(0,\sigma^2\mathbb{E}\left[X'X\right]^{-1}\right)\end{aligned}$$

So, we would approximate this distribution as

$$\widehat{\boldsymbol{\beta}} \overset{a}{\sim} \mathbb{N}(\boldsymbol{\beta},\widehat{\sigma}^2(X'X)^{-1}) = \mathbb{N}\left(\boldsymbol{\beta},\text{se}\left[\widehat{\boldsymbol{\beta}}\right]\right)$$

as $\widehat{\sigma}^2$ and $(X'X)^{-1}$ are consistent for σ^2 and $\mathbb{E}[X'X]^{-1}$. $\square$
Hooray!

T-Statistic

Question 2

So, we know under MLR Assumptions 1-5 that the OLS estimator is asymptotically normal even when dividing by its standard error. Is it invalid to use the t-distribution for inference then?

T-Statistic

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Answer to Question 2

No. It's just as valid to use the t-distribution for computing critical and p-values because the t-distribution converges to the standard normal as its degrees of freedom gets large, i.e., $n - k$ gets large which happens when n gets large which means our OLS estimator is approximately standard normal! You can use either and get reliable results, but for sufficiently large samples we usually use the standard normal distribution.

Sufficiently Large

Question 3

I've said sufficiently large n throughout this lecture. What exactly does that mean?

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Answer to Question 3

To be comfortable replacing the standard normal distribution with the t-distribution for inference, I would recommend having at least fifty degrees of freedom. Take a look at the R notes for some justification.

OLS Asymptotic Efficiency

Theorem 3: OLS Asymptotic Efficiency

Under MLR Assumptions 1-5, the OLS estimator is asymptotically efficient: $\text{avar}\left[\widehat{\boldsymbol{\beta}}\right] \leq \text{avar}\left[\widetilde{\boldsymbol{\beta}}\right]$ for any other unbiased estimator $\widetilde{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$.

- We need MLR Assumption 5 of homoskedasticity just as in the regular Gauss-Markov Theorem.

Thank You!